Non-Gaussianity and multi-field models of inflation

Matteo Biagetti

Università di Padova
Dipartimento di Fisica "G. Galilei"

14 Dicembre 2011

Relatore: Dr. Nicola Bartolo
Correlatore: Prof. Sabino Matarrese

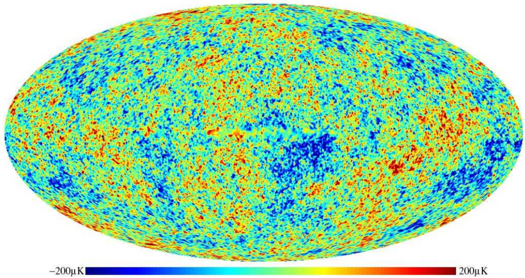


Figure: The Cosmic Microwave Background: data from WMAP satellite.

- Extremely homogeneus and isotropic temperature distribution;
- Small anisotropies are present of order $\Delta T/T \sim 10^{-5}$.

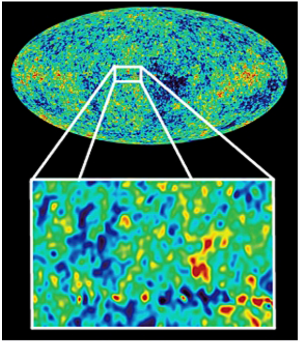
Before the last scattering of the photons, the universe was *opaque* $\Longrightarrow$ the CMB is the earliest source of information about the past history of the universe.

1. How do we study the early universe?
   - The Cosmic Microwave Background (CMB)
   - What is the source of the anisotropies?
2. The inflationary scenario
   - What is inflation?
   - Quantum fluctuations from inflation
3. Non-Gaussianity from inflation
   - The statistical properties of the primordial perturbations
   - Constraint relations between f_{NL} and τ_{NL} in multi-field models
4. Conclusions and future prospects
   - Accounting for the non-Gaussianities at horizon crossing
   - The WMAP and Planck experiments

The structure problem

The cosmic structure nowadays (galaxies, clusters of galaxies, etc.) came from density inhomogeneities in the early universe.
We see those same inhomogeneities in the CMB.

- How did those inhomogeneities get there?
- Why are they just the right magnitude and size to produce the structures we see today?



The idea of accelerated expansion

Inflation is defined as an epoch of accelerated expansion of the early Universe which occurred at energies up to $\sim 10^{16}$ GeV.

The accelerated expansion is driven by a scalar field, the *inflaton*, which slowly rolls down the minimum of its potential during the inflationary stage.

The total energy density is dominated by the vacuum energy of the field, giving the Friedmann equation

$$H^2 \simeq \frac{8\pi G}{3} V(\langle\phi\rangle).$$

The value of H is therefore almost constant during the expansion and the scale factor grows exponentially $a(t) \propto e^{Ht}$.

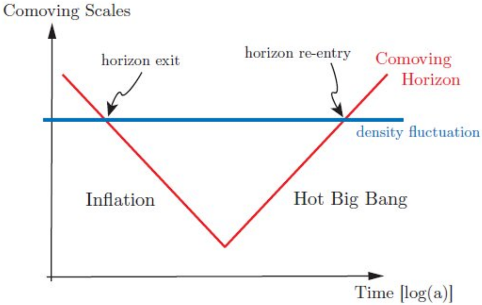
1. How do we study the early universe?
   - The Cosmic Microwave Background (CMB)
   - What is the source of the anisotropies?
2. The inflationary scenario
   - What is inflation?
   - Quantum fluctuations from inflation
3. Non-Gaussianity from inflation
   - The statistical properties of the primordial perturbations
   - Constraint relations between f_{NL} and τ_{NL} in multi-field models
4. Conclusions and future prospects
   - Accounting for the non-Gaussianities at horizon crossing
   - The WMAP and Planck experiments

The solution to the structure problem

One can separate the scalar field into two contributions:

$$\phi(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t).$$

- A given mode of fluctuation will start within the comoving cosmological horizon $(aH)^{-1}$.
- The mode grows much more rapidly than the comoving horizon and thus soon exits it.
- At this point the microphysical processes which occurred at the beginning of accelerated expansion are "frozen".



Once inflation is over, the mode reenters the comoving horizon at some stage after inflation, perturbing photon and baryons to produce the inhomogeneities we see in the CMB.

1. How do we study the early universe?
   - The Cosmic Microwave Background (CMB)
   - What is the source of the anisotropies?
2. The inflationary scenario
   - What is inflation?
   - Quantum fluctuations from inflation
3. Non-Gaussianity from inflation
   - The statistical properties of the primordial perturbations
   - Constraint relations between f_{NL} and τ_{NL} in multi-field models
4. Conclusions and future prospects
   - Accounting for the non-Gaussianities at horizon crossing
   - The WMAP and Planck experiments

Why do we study non-Gaussianity in inflationary models?

If it is true that primordial perturbations are left imprinted in the CMB, any primordial deviation from a Gaussian distribution has to be observed in the anisotropies in the CMB. $\Longrightarrow$ *we can test the various models of inflation by predicting their level of non-Gaussianity.*

The statistical properties of the primordial perturbations are described via Gauge-invariant quantities.

The curvature perturbation on uniform energy density hypersurfaces ζ is frequently used and it represents intuitively the density perturbations $\delta\rho/\rho$.

We define the power-spectrum, bispectrum and trispectrum (2, 3 and 4-point correlation functions in Fourier space) of ζ:

$$\begin{aligned}\langle\zeta(\mathbf{k_1})\zeta(\mathbf{k_2})\rangle &= (2\pi)^3\delta^3(\mathbf{k_1}+\mathbf{k_2})P_\zeta(k),\\ \langle\zeta(\mathbf{k_1})\zeta(\mathbf{k_2})\zeta(\mathbf{k_3})\rangle &= (2\pi)^3\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})B_\zeta(k_1,k_2,k_3),\\ \langle\zeta(\mathbf{k_1})\zeta(\mathbf{k_2})\zeta(\mathbf{k_3})\zeta(\mathbf{k_4})\rangle &= (2\pi)^3\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4})T_\zeta(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3},\mathbf{k_4}).\end{aligned}$$

B_ζ and T_ζ measure the level of non-Gaussianity of the model and to connect them to measurable quantities one defines non-linearity parameters f_{NL} and τ_{NL} which are proportional to their amplitudes:

$$\begin{aligned}B_\zeta(k_1,k_2,k_3) &= \frac{6}{5}f_{NL}(P_\zeta(k_1)P_\zeta(k_2)+2\ perms),\\ T_\zeta(k_1,k_2,k_3,k_4) &= \frac{1}{2}\tau_{NL}(P_\zeta(k_1)P_\zeta(k_2)P_\zeta(|\mathbf{k_1}-\mathbf{k_4}|)+23\ perms).\end{aligned}$$

One can write ζ as an expansion in the fluctuations $\delta\phi$ (δN formalism)

$$\zeta_f(\mathbf{x}) = \sum_{n=1}^{\infty} (\delta\phi_i)^n \left(\frac{\partial}{\partial\phi_i}\right)^n N(t_f, t_i),$$

being $N(t_f, t_i) = \int_{t_i}^{t_f} H(\phi) dt$ the number of e-foldings.

Thus the N-point functions of ζ can be written as a sum of various terms proportional to the ones of the fluctuations $\delta\phi$

$$\begin{aligned}
\langle \delta\phi^A_{\mathbf{k_1}} \delta\phi^B_{\mathbf{k_2}} \rangle &= (2\pi)^3 \delta^3(\mathbf{k_1} + \mathbf{k_2}) C^{AB}(k); \\
\langle \delta\phi^A_{\mathbf{k_1}} \delta\phi^B_{\mathbf{k_2}} \delta\phi^C_{\mathbf{k_3}} \rangle &= (2\pi)^3 \delta^3(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3}) B^{ABC}(k_1, k_2, k_3); \\
\langle \delta\phi^A_{\mathbf{k_1}} \delta\phi^B_{\mathbf{k_2}} \delta\phi^C_{\mathbf{k_3}} \delta\phi^D_{\mathbf{k_4}} \rangle_c &= (2\pi)^3 \delta^3(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3} + \mathbf{k_4}) T^{ABCD}(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}, \mathbf{k_4}).
\end{aligned}$$

where $A, B, C, D = 1, 2, ..., n$ are summed indices using the Einstein convention, being n the total number of fields contributing to inflation.

1. How do we study the early universe?
   - The Cosmic Microwave Background (CMB)
   - What is the source of the anisotropies?
2. The inflationary scenario
   - What is inflation?
   - Quantum fluctuations from inflation
3. Non-Gaussianity from inflation
   - The statistical properties of the primordial perturbations
   - Constraint relations between f_{NL} and τ_{NL} in multi-field models
4. Conclusions and future prospects
   - Accounting for the non-Gaussianities at horizon crossing
   - The WMAP and Planck experiments

In the case of standard single-field models of inflation, *the non-linearity parameter* $|f_{NL}| \ll 1$ *and thus the distribution of primordial perturbations is almost Gaussian.*

A relation between f_{NL} and τ_{NL} has been recently demonstrated for a class of multi-field models†

$$\tau_{NL} \geq \frac{1}{2}\left(\frac{6}{5} f_{NL}\right)^2 ,$$

If this inequality were violated observationally, the multi-field models considered would be ruled out as a possible mechanism for the generation of primordial perturbations.

Finding constraint relations between f_{NL} *and* τ_{NL} *is therefore important to discriminate among the various models of inflation.*

† *N. S. Sugiyama, E. Komatsu, T. Futamase, Non-Gaussianity Consistency Relation for Multi-field Inflation, Phys. Rev. Lett.* **106***, 251301 (2011).*

The hypothesis of Gaussian fields at horizon crossing

It is thus very important to understand the level of validity of this constraint relation.

The inequality has been demonstrated under the following assumptions:

- Scalar fields are responsible for generating curvature perturbations;
- Fluctuations in scalar fields at the horizon crossing are Gaussian;
- The expansion written by virtue of the δN formalism is truncated to the fourth order in the fluctuations $\delta\phi_i^4$.

Non-Gaussianities can be generated when modes of fluctuation of different fields have similar wavelenghts and exit the horizon at about the same time.

Instead, with the second hypothesis one assumes that these non-Gaussianities are negligible.

In fact, *if the fluctuations $\delta\phi$ are Gaussian at the horizon crossing, the 3 and (connected) 4-point functions of $\delta\phi$ are zero and the statistical properties of the fluctuations $\delta\phi$ are determined only by their power-spectrum.*

$$\begin{aligned} C^{AB}(k) &= \delta^{AB} P(k) = \delta^{AB} \frac{H_*^2}{2k^3} \\ B^{ABC}(k_1, k_2, k_3) &= 0 \\ T^{ABCD}(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}, \mathbf{k_4}) &= 0 \end{aligned}$$

In my thesis, I relaxed this hypothesis and thus I admit a non-zero value for B^{ABC} and T^{ABCD}, accounting for possible interactions among the fields at horizon crossing.

A diagrammatic approach

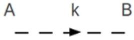
The method used to calculate the N-point correlation functions is based on an analogy with the Feynman rules of Quantum Field Theory and it was first introduced by D.Wands et al.[†].

One is able to write appropriate Feynman rules and draw the diagrams which represent in a very direct and simple way the various terms contributing to the correlation functions.

Using this formalism, *I found new and more general expressions for f_{NL} and τ_{NL}, accounting for possible large non-Gaussianities at horizon crossing.*

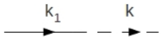
[†] *C. Byrnes, K. Koyama, M. Sasaki, D. Wands, Diagrammatic Approach to non-Gaussianity from Inflation, JCAP* **0711**, *027 (2007).*

One can think of $C^{AB}(k)$ as the scalar field propagator



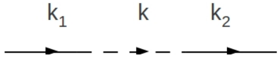
$C^{AB}(k)$

and N_A, N_B the vertex factors



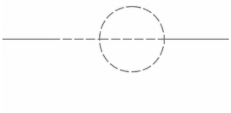
$(2\pi)^3 \delta^{(3)}(k_1 - k) N_A$

Thus one can draw the diagram for the tree-level term of the power-spectrum,



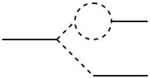
$$P_\zeta(k) = N_A N_B C^{AB}(k).$$

Some example:



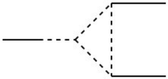
$$\frac{1}{3} N_A N_{BCD} \int \frac{d^3 q_1 d^3 q_2}{(2\pi)^6} T^{ABCD}(\mathbf{q_1}, \mathbf{q_2} - \mathbf{q_1}, \mathbf{k_1} - \mathbf{q_2}, \mathbf{k_2})$$

Power-spectrum, 2-loop diagram



$$\frac{1}{2} N_{AB} N_{CD} N_E C^{AE}(|\mathbf{k_1} + \mathbf{k_2}|) \int \frac{d^3 q}{(2\pi)^3} B^{BCD}(q, k_2, |\mathbf{q} - \mathbf{k_2}|)$$

Bispectrum, 1-loop diagram



$$\frac{1}{2}N_{ABC}N_D N_E \int \frac{d^3q}{(2\pi)^3} C^{AE}(|\mathbf{k_1}+\mathbf{k_2}|)B^{BCD}(q,|\mathbf{q}-\mathbf{k_2}|,k_2)$$

Bispectrum, 1-loop diagram

$$\frac{1}{2}N_{AB}N_C N_{DEFG}N_H \int \frac{d^3q}{(2\pi)^3} C^{AF}(|\mathbf{k_1}-\mathbf{q}|)C^{BE}(q)C^{CD}(k_2)C^{GH}(k_4)$$

Trispectrum, 1-loop diagram

1. How do we study the early universe?
   - The Cosmic Microwave Background (CMB)
   - What is the source of the anisotropies?
2. The inflationary scenario
   - What is inflation?
   - Quantum fluctuations from inflation
3. Non-Gaussianity from inflation
   - The statistical properties of the primordial perturbations
   - Constraint relations between f_{NL} and τ_{NL} in multi-field models
4. Conclusions and future prospects
   - **Accounting for the non-Gaussianities at horizon crossing**
   - The WMAP and Planck experiments

My thesis dealt with the problem of the prediction of non-Gaussianities in the primordial perturbations for multi-field models of inflation.

- I calculated the power-spectrum, bispectrum and trispectrum of the curvature perturbation ζ accounting for possible large non-Gaussianities at horizon crossing, generalizing the results found in a previous paper.
- A diagrammatic approach has been used, extending a formalism introduced before with the calculation of new 1-loop bispectrum and trispectrum diagrams.
- These expressions are useful to demonstrate, once an appropriate model is chosen, that the inequality

$$\tau_{NL} \geq \frac{1}{2}\left(\frac{6}{5} f_{NL}\right)^2,$$

  may be violated.
- A new constraint relation can be found between f_{NL} and τ_{NL} in the equilateral limit.

1. How do we study the early universe?
   - The Cosmic Microwave Background (CMB)
   - What is the source of the anisotropies?
2. The inflationary scenario
   - What is inflation?
   - Quantum fluctuations from inflation
3. Non-Gaussianity from inflation
   - The statistical properties of the primordial perturbations
   - Constraint relations between f_{NL} and τ_{NL} in multi-field models
4. Conclusions and future prospects
   - Accounting for the non-Gaussianities at horizon crossing
   - The WMAP and Planck experiments

The 7-year WMAP observations

To date, the data we have for the non-linearity parameter f_{NL} come from the 7-year WMAP observations†:

	f_{NL}^{local}	f_{NL}^{equil}
WMAP	32 ± 21	26 ± 140

Table: Estimates and the corresponding 68% intervals of the primordial non-Gaussianity parameters f_{NL}^{local} and f_{NL}^{equil}.

† *E. Komatsu et al., Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological interpretation, ApJS* **192**, *18 (2011).*

The Planck satellite

The Planck satellite was launched in May 2009. The results for the observations of the CMB anisotropies will be publicly available soon. The most reliable forecast on f_{NL} and τ_{NL} for Planck are[††] (minimum error bars)

	f_{NL}	τ_{NL}
Planck	8	1550

Table: The minimum error bars at 1σ for f_{NL} and τ_{NL} for the Planck experiment.

[††] *J. Smidt et al., CMB Constraints on Primordial non-Gaussianity from the Bispectrum (f_{NL}) and Trispectrum (g_{NL} and τ_{NL}) and a New Consistency Test of Single-Field Inflation, Phys. Rev.* **D 81**, *Issue 12, id. 123007 (2011).*

Gaussian distributions

What do we mean by "Gaussian distribution"?

Let us consider the distribution of temperature anisotropy of the CMB that we observe on the sky, $\Delta T(\hat{\mathbf{n}})$. The temperature anisotropy is Gaussian when its probability density function (PDF) is given by

$$P(\Delta T) = \frac{1}{(2\pi)^{N_{pix}/2}|\xi|^{1/2}} e^{-\frac{1}{2}\sum_{ij}\Delta T_i(\xi)^{-1}_{ij}\Delta T_j}$$

where $\Delta T_i = \Delta T(\hat{\mathbf{n}}_i)$, $\xi_{ij} = \langle \Delta T_i \Delta T_j \rangle$ is the covariance matrix (or the 2-point correlation function) of the temperature anisotropy.

A Gaussian distribution is completely determined by its 2-point correlation function ξ_{ij}.

Non-Gaussianity

Any deviation from a Gaussian distribution is called *non-Gaussianity*.

When fluctuations in the CMB are non-Gaussian, the 2-point correlation function is no more sufficient to characterize the distribution, one has to calculate higher order correlation functions.

In fact, when non-Gaussianity is weak, one may expand the PDF around a Gaussian distribution and take the first terms of the expansion, which depend on the 3 and 4-point functions.

The observational tests of non-Gaussianity

The first step, from the observational point of view, is to introduce the angular N-point correlation function, for example, of the temperature anisotropies $\Delta T(\hat{\mathbf{n}})$

$$\langle \Delta T(\hat{\mathbf{n}}_1)\Delta T(\hat{\mathbf{n}}_2)...\Delta T(\hat{\mathbf{n}}_N)\rangle,$$

where the brackets denotes ensemble average,

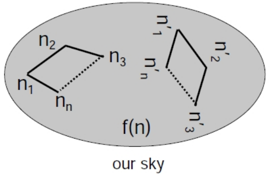


Figure: A schematic view of the statistic isotropy of the angular correlation function: as long as the isotropy is preserved, we can average $\Delta T(\hat{\mathbf{n}}_i)$ over all possible orientations and positions on the sky.

It is however more convenient to espand each $\Delta T(\hat{\mathbf{n}})$ into spherical harmonics, the orthonormal basis on the sphere, as

$$\Delta T(\hat{\mathbf{n}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} a_{lm} Y_{lm}(\hat{\mathbf{n}}),$$

thus considering the ensemble average in the coefficients

$$\langle a_{l_1 m_1} a_{l_2 m_2} \dots a_{l_N m_N} \rangle,$$

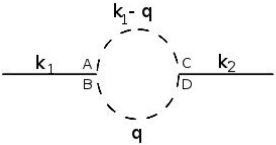
which is called *angular N-point harmonic spectrum*.

In fact, we are not able to measure the ensemble average, but one particular realization $a_{l_1 m_1} a_{l_2 m_2} \dots a_{l_N m_N}$, which is so noisy that some kind of average is needed to reduce the noise.

One chooses the average to be over an azimuthal orientation on the sky m_i, provided that it is made with an appropriate weight, thus reducing the statistical error of the measured harmonic spectra.

1-loop diagrams

In the calculation of N-point function for ζ, 1-loop diagrams have been considerated. These contributions diverge in general. For example, one can take a 1-loop diagram for the power-spectrum of ζ



$$P_\zeta = \frac{1}{2} N_{AB} N_{CD} \int \frac{d^3 q_1}{(2\pi)^3} C^{AB}(q_1) C^{CD}(|\mathbf{k_1} - \mathbf{q_1}|)$$

Specifically, the divergence is of the form

$$\int_{L^{-1}} \frac{d^3 q}{(2\pi)^3} \frac{1}{|\mathbf{q}|^3} \frac{1}{|\mathbf{q} - \mathbf{k}|^3} = \frac{1}{k^3} \ln(kL)$$

where the subscript L^{-1} indicates that the integrand is set equal to zero in a sphere of radius L around each singularity and the discussion makes sense only for $kL \gg 1$.